NEAR-SYMPLECTIC 6–MANIFOLDS

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ABSTRACT. We give a generalization of the concept of near-symplectic structures to 6-manifolds M^6. We show that for some contact 3-manifold Z, then $Z^3 \times S^2$ admits a contact structure that is PS-overtwisted, and that appears naturally in our context. Moreover, there is a contact fibration $Z \times S^2 \setminus C \to S^2 \setminus C$ outside a 1-submanifold C. According to our definition, a closed 2-form on M^6 is near-symplectic, if it is symplectic outside a 3-submanifold Z, where ω^2 vanishes transversally. We depict how this notion relates to near-symplectic 4-manifolds and BLFs via some examples. We define a generalized BLF as a singular map with indefinite folds and Lefschetz-type singularities, showing that with this map a 6-manifold carries such a near-symplectic structure. This setting is applied to detect a PS-overtwisted contact structure on $Z \times S^2$ and then a contact fibration over an open dense subset of S^2.

1. Introduction

Near-symplectic 4-manifolds are equipped with a closed 2-form that is non-degenerate outside disjoint union of circles, where it vanishes. These structures where studied in detail in the work of Auroux, Donaldson and Katzarkov [ADK05] using broken Lefschetz fibrations (BLFs). It was shown that there is a direct correspondance between BLFs and near-symplectic 4-manifolds. These results extended the theorems of Donaldson [Don99] and Gompf [GS99] on Lefschetz fibrations and symplectic manifolds, which in turn expanded Thurston's theorem on symplectic fibrations [Thu76]. Broken Lefschetz fibrations have found fruitful application in low-dimensional topology, for example in holomorphic quilts [WW] and Lagrangian matching invariants [Per07, Per08]. A relevant existence result states that every smooth closed oriented 4-manifold admits a BLF [GK07, Bay09, Lek09, AK08]. Furthermore, on the contact side, we find that near-symplectic 4-manifolds induce an overtwisted contact structure on a 3-submanifold, as found by Gay and Kirby [GK04].

In contact topology we find an asymmetrical cartography between lower and higher dimensions. Whereas the classification of contact 3–manifolds has been successfully understood, less is known about the panorama in dimensions 5 and above. In recent years, the notion of *overtwistedness* in higher dimensions has been explored in more detail [Nie06, EP11, MNW12, Mor09]. Niederkrüger proposed the *plastikstufe* as a generalization of the overtwisted disk via a n-submanifold with a particular foliation inside a $(2n-1)$–contact manifold [Nie06]. This definition is justified by the fact that if a contact manifold of dimension greater or equal to 5 admits a plastikstufe, then it is non-fillable. Etnyre and Pancholli have shown

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that a manifold admitting a contact structure also admits an overtwisted one by a generalization of the Lutz twist [EP11]. More recently, Massot, Niederkrüger, and Wendl have extended the notions of weak and strong fillability to higher dimensions using a more refined version of the plastikstufe, called *bLob* or *bordered Legendrian open book* [MNW12].

This work aims to find a good notion to generalize near-symplectic structures on higher dimensions, and focuses on the 6-dimensional case. It is also one of our goals to display the presence of PS-overtwistedness in a setting different from a question of fillability. In this case it is under the near-symplectic framework. We also want to present a construction of a contact fibration over an open dense subset of S^2 using this ambient. Now we present the outline of this paper.

In section 2.1, we suggest a definition of a near-symplectic structure in dimension 6. The idea is to relax the non-degeneracy condition of the symplectic form, while preserving some kind of self-duality of the 2-form. On a smooth, orientable, 6-manifold, we call closed 2-forms near-symplectic, if they degenerate to rank 2 at a 3-submanifold Z_ω, but are symplectic everywhere else. Examples of near-symplectic 6-manifolds are given in sections 2.2 and 3.2. The latter depicts the cases of near-symplectic 6-manifolds coming from 4-dimensional ones. The interaction between near-symplectic structures in these two dimensions illustrates how the near-symplectic concept on 6-manifolds relates itself naturally with the one on 4-dimensions.

Next, we study the question of the existence of these structures using singular fibrations, analogous to BLFs. We define a *generalized BLF* as a submersion $f\colon M^6 \to X^4$ with two types of singularities: 2-submanifolds of extended Lefschetz type singularities locally modeled by complex coordinate charts $(z_1, z_2, z_3) \mapsto (z_1, z_2^2 + z_3^2)$, and indefinite fold singularities contained in 3-submanifolds modeled by real charts $(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, -x_1^2 + x_2^2 + x_3^2)$. We denote by Σ the set of fold singularities of f. For more details see definition 3.2. We state our first result.

Theorem 1. *Let $f\colon M^6 \to (X^4, \omega)$ be a generalized BLF from a smooth closed oriented 6-manifold M to a compact symplectic 4-manifold X. Assume that there is a class $\alpha \in H^2(M)$, such that it pairs positively with every component of every fibre, and that the restriction of α to the singularity set of folds, Σ, is the zero element in $H^2(\Sigma)$. Then, there is a near-symplectic form ω on M^6, with singular locus Z_ω equal to Σ, and symplectic fibres outside Σ.*

The proof appearing section 3.3 constructs an explicit closed 2-form that vanishes at the singularity set of the mapping, and then it pulls back the symplectic form of the base by modifying the singularity set of f in a symplectic manner. Afterwards, everything is glued together into a global 2-form. This statement follows a similar line of reasoning as Auroux-Donaldson-Katrzarkov [ADK05] construction of near-symplectic forms using BLFs. In the last two sections, we explore the interplay with contact structures. We study this at the level of a 5-submanifold. First, we show how the contact structure is obtained from the near-symplectic form.

Proposition 1. *Consider the standard map of a generalized BLF with singularities modeled as in definition 3.2, $f\colon Z \times \mathbb{R}^3 \to (Z \times \mathbb{R}, \omega = d(e^t\alpha))$, where $(Z \times \mathbb{R})$ denotes the symplectization of Z. The near-symplectic form of $(Z \times \mathbb{R}^3, \omega_{\text{ns}})$ induces a contact structure on $Z \times S^2$.*

We give the proof in section 4.1. By induced contact structure, we understand that ξ comes directly from the near-symplectic form. We build a vector field Y pointing in the normal direction of the bundle to obtain a 1-form $\alpha_N := \iota_Y \omega_{\text{ns}}$. This 1-form together with the pullback of some contact form of the 3-dimensional singular set Z defines a contact form on $Z \times S^2$. In the last section 5 we classify the type of contact structure.

Theorem 2. *Given a contact 3-manifold Z^3, there is a PS-overtwisted contact structure on $Z^3 \times S^2$ induced from a near-symplectic manifold. Furthermore, there is a 1-dimensional submanifold C of S^2 such that the natural projection $\pi\colon (Z^3 \times S^2) \setminus C \to S^2 \setminus C$ is a contact fibration with the induced contact structure.*

It is known that any contact manifold can be given a PS-overtwisted contact structure by work of [Pre07, NvK07] and [EP11]. However, we show that this property appears naturally in our context using a different method. The proof appears in section 5.2. The property of ξ to be PS-overtwisted can be detected in a rather direct way by its contact form α. We consider the cotangent bundle $T^*Z \simeq Z \times \mathbb{R}^3$ with near-symplectic structure together with a generalized BLF mapping down to the symplectization $(Z \times \mathbb{R}, \omega)$. Then we look at the induced contact structure on the unit cotangent bundle (ST^*Z_Z, α). We obtain a decomposition of $ST^*Z \simeq Z \times S^2$ in 2 Giroux domains $\Sigma_\pm \times S^1$ by looking at the dividing set of α. This allows us to frame it in the context of *bLobs* to perform a blowing down operation. Then, a result from Massot-Niederkruger-Wendl [MNW12] tells us that one of these components is enough for $Z \times S^2$ to be PS-overtwisted.

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2. **Near-symplectic forms**

We first recall the definition of near-symplectic forms on 4-manifolds.

Definition 2.1. Let X be a smooth oriented 4-manifold. Consider a closed 2-form $\omega \in \Omega^2(X)$ such that $\omega^2 \geq 0$ and such that ω_p only has rank 4 or rank 0 at any point $p \in X$, but never rank 2. The form ω is called *near-symplectic*, if it is non-degenerate or it vanishes transversally along circles. That is, for every $p \in X$, either

(1) $\omega_p^2 > 0$, or
(2) $\omega_p = 0$, and $\text{Rank}(\nabla \omega_p) = 3$, where $\nabla \omega_p \colon T_pX \to \Lambda^2 T_p^* X$ denotes the intrinsic gradient of ω.

It follows from the condition on $\nabla\omega_p$ that the singular locus Z_ω is a smooth 1-submanifold of X [ADK05], [Per06]. A prototypical example of a near-symplectic 4-manifold is given by $X = S^1 \times Y^3$, where Y is a closed 3-manifold. Consider a closed 1-form $\beta \in \Omega^1(Y)$ with indefinite Morse critical points and let t be the parameter of S^1. The 2-form $\omega = dt \wedge \beta + *(dt \wedge \beta)$ is near-symplectic, where the Hodge $*$-operator is defined with respect to the product metric on S^1 and Y. The singular locus $Z_\omega = \{p \in X \mid \omega_p = 0\}$ is in this case $S^1 \times \text{Crit}(\alpha)$.

2.1. **Near-symplectic 2n-manifolds**

The following exposition is closely based on a message from Tim Perutz. Let V be an oriented $2n$-manifold, and $\omega \in \Omega^2(V)$ a closed 2-form such that $\omega^n \geq 0$ everywhere. Suppose that at some point p, the kernel K has dimension 4.

$$K = \{v \in T_pV \mid \omega_p(v, \cdot) = 0\}$$

We have an intrinsic gradient $\nabla\omega \colon K \to \Lambda^2 T_p^* V$. We can restrict this map to bivectors in K to get a map

$$D_K \colon K \to \Lambda^2 K^* \tag{1}$$

Then the wedge square gives us a non-degenerate quadratic form $q\colon \Lambda^2 K^* \to \Lambda^4 K^*$.

Proposition 2.2. *The image* $\text{Im}(D_K)$ *is a semi-definite subspace of* $\Lambda^2 K^*$ *with respect to the wedge square form.*

Proof. Take an arbitrary tangent vector $v \in T_pV$ and choose coordinates such that p is the point at the origin. By our assumption on ω, we have $\omega^n(t \cdot v) \geq 0$ for all scalars t. Yet, if we use a Taylor expansion to write $\omega(t \cdot v) = \omega(0) + t \cdot \nabla_0\omega(v) + O(t^2)$ and take $v \in K$, we have

$$\omega^n(t \cdot v) = \binom{n}{2} \cdot t^2 \cdot \omega(0)^{n-2} \wedge (\nabla\omega(v))^2 + O(t^3)$$

from which the claim follows. $\square$

Recalling that the wedge product splits $\Lambda^2\mathbb{R}^4$ into a positive and negative subbundles each of dimension 3, then the image of D_K has dimension at most 3. With this in mind we come to our defintion.

Definition 2.3. The 2-form $\omega \in \Omega^2(V^{2n})$ is *near-symplectic*, if it is closed, $\omega^n \geq 0$, and at a point p where $\omega^n = 0$, one has that the kernel K is 4-dimensional and that the $\text{Im}(D_K)$ has dimension 3.

Then the map D_K has 1-dimensional kernel. If we look at ω^{n-1} then it vanishes at p, and $G = \nabla\omega^{n-1}(p)$ is intrinsically defined. If we choose coordinates (x_k) so that K is defined by the vanishing of all but the last four dx_k, we have

$$G = (n-1)\omega(p)^{n-2}\nabla_p\omega$$

where the gradient on the right is defined using the coordinates. The 2-form ω is symplectic on the submanifold Z where the last 4 coordinates are zero. We

can adjust the coordinates to Darboux form, so that ω is constant on Z. Hence $\nabla_p\omega(\partial x_i) = 0$ for $i = 1, \dots, 2n-4$. However,

$$\ker G = \ker(\nabla_p\omega)$$

and now one sees that this is a codimension 3 subspace containing the line $\ker(D_K)$. Hence the degeneracy locus Z of the near-symplectic form is a codimension 3-submanifold of V^{2n}.

Remark. The property of $\omega|^n_{V\setminus Z} > 0$ guarantees that the whole V^{2n} is orientable. This is due to the fact that Z is a submanifold of codimension 3. In fact, it follows from a standard algebraic topological argument that this orientability property is true on any dimension if the codimension of the submanifold is greater or equal to two. That is to say, if ω is a 2-form on a $2n$-manifold V, K is a k-dimensional submanifold of V, and $\omega^n > 0$ on $V \setminus K$, then V is oriented if codim$(K) \geq 2$.

Remark. In dimension 4, near-symplectic structure are related to self-dual harmonic forms. An obvious obstacle in dimensions 6 and above is that there is no analogue of self-duality for 2-forms. Some local models of near-symplectic forms on 6-manifolds M^6 seem to indicate that near-symplectic forms could be equivalent to $\omega = *\omega^2$ for some generic metric, outside the singular locus Z.

2.2. **Examples**

1. Let $M^6 = N^3 \times Y^3$, where N and Y are closed, connected, orientable, smooth 3-manifolds. If N fibers over S^1, we can get a nowhere vanishing closed 1-form $\beta \in \Omega^1(N)$, and it is transitive. With very little work the transitivity can be checked. This implies that it is intrinsically harmonic. Let $\alpha \in \Omega^1(Y)$ be a closed 1-form with indefinite Morse singular points. By Calabi's and Ko Honda's [Hon04], [Hon99] theorems this form can be replaced by an intrinsically harmonic 1-form lying in the same cohomology class and having the the same Morse numbers. Equip the 6-manifold with the following 2-form:

$$\omega = \beta \wedge \alpha + (*_N\beta) + (*_Y\alpha)$$

This 2-form is closed, $\omega^3 \geq 0$ on M, and the singular locus where $\omega^2 = 0$ is at $N^3 \times$ Crit(α), thus near-symplectic. Using local coordinate charts, the transversality condition on the intrinsic gradient can be seen on the (6×15)-matrix coming from the linearization $D\omega^2$ of $\nabla\omega^2$, which has rank 3 at the singular points.

2. On $\mathbb{R}^6$ with standard coordinates $(t_1, t_2, t_3, x_1, x_2, x_3)$:

$$\begin{aligned}\omega &= dt_1 \wedge dt_2 - 2x_1(dt_3 \wedge dx_1 + dx_2 \wedge dx_3)\\ &\quad + x_2(dt_3 \wedge dx_2 - dx_1 \wedge dx_3) + x_3(dt_3 \wedge dx_3 + dx_1 \wedge dx_2)\end{aligned}$$

The singular locus of $\omega^2 = 0$ is given by $Z_\omega = \left\{p \in \mathbb{R}^6 \mid x_1 = x_2 = x_3 = 0\right\}$ and $\omega^3 > 0$ away from Z_ω.

3. Fibrations

3.1. Near-symplectic fibrations

We call up the definition of broken Lefschetz fibrations on four dimensions. On a smooth, closed 4-manifold X, a *broken Lefschetz fibration* or *BLF* is a smooth map $f\colon X^4 \to S^2$ from a closed 4-manifold X^4 to S^2 with two types of singularities:

(1) isolated *Lefschetz-type* singularities, contained in the finite subset of points $B \subset X^4$, which are locally modeled by complex charts
$$\mathbb{C}^2 \longrightarrow \mathbb{C} \quad , \quad (z_1, z_2) \longmapsto z_1^2 + z_2^2$$
(2) *indefinite fold* singularities, also called *broken*, contained in the smooth embedded 1-dimensional submanifold $\Gamma \subset X^4 \backslash B$, which are locally modelled by the real charts
$$\mathbb{R}^4 \longrightarrow \mathbb{R}^2 \quad , \quad (t, x_1, x_2, x_3) \longmapsto (t, x_1^2 + x_2^2 - x_3^2)$$

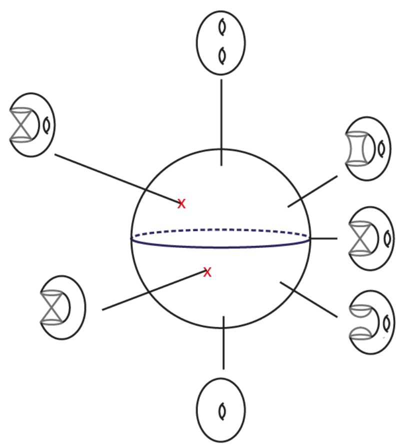

FIGURE 1. Example of a BLF with 1 circle of folds and 2 Lefschetz points

In [ADK05] these mappings were studied under the name of singular Lefschetz pencils. It was shown that there is a relation between BLFs and near-symplectic manifolds. Up to blow ups, a near-symplectic 4-manifold X can be decomposed into a BLF. The other direction is given by the following theorem.

Theorem 3.1 ([ADK05]). *Given a BLF with singularity set $\Gamma \sqcup B$ on a closed oriented 4-manifold X, with the property that there is a class $\alpha \in H^2(X)$, such that it pairs positively with every component of every fiber, then X carries a near-symplectic structure with zero-locus being equal to the set of broken singularities of f.*

Our theorem 1 shows a similar statement on near-symplectic 6-manifolds. Now we define a map that will play an analogous role of a BLF two dimensions higher. This map is a submersion with folds and Lefschetz-type singularities. Notice that submersion with folds are stable if the map f restricted to its fold set is an immersion with normal crossings [GG73]. By stable we mean that any nearby map $\tilde{f}$ is identical to f after changes of coordinates.

Definition 3.2. Let M be a smooth, closed 6-manifold M. By a *generalized broken Lefschetz fibration* we mean a submersion to a smooth oriented 4-manifold $f\colon M^6 \to X^4$ with two type of singularities:

(1) ”extended” *Lefschetz-type* singularities, locally modelled by

$$\mathbb{C}^3 \to \mathbb{C}^2$$
$$(z_1, z_2, z_3) \to (z_1, z_2^2 + z_3^2)$$

These singularities are contained in 2-submanifolds cross a Lefschetz singular point. Singular fibres present an isolated nodal singularity, but nearby fibres are smooth and convex.

(2) *indefinite fold* singularities, locally modeled by

$$\mathbb{R}^6 \to \mathbb{R}^4$$
$$(t_1, t_2, t_3, x_1, x_2, x_3) \mapsto (t_1, t_2, t_3, -x_1^2 + x_2^2 + x_3^2)$$

The fold locus is an embedded 3-dimensional submanifold, and we denote its set by Σ. Singular fibres have again a nodal singularity, but this time crossing Σ changes the genus of the regular fibre by one.

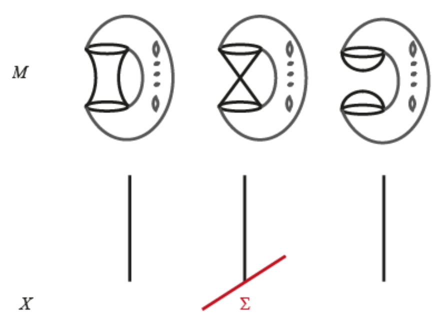

FIGURE 2. Fibres: regular (left, right) and singular (middle)

If we consider the total space to be near-symplectic, then we will refer to the previous map as a *near-symplectic fibration*.

3.2. **Examples**

3.2.1. **Pullback bundle**

We can obtain examples of near-symplectic 6-manifolds and near-symplectic fibrations via a pullback bundle construction. In the above diagram, let M and X be oriented, closed, 4-manifolds, f and g smooth, and $W = \{(x, m) \in X \times M \mid f(x) = g(m)\}$.

$$\begin{array}{ccc} W & \xrightarrow{\tilde{f}} & M^4 \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X^4 & \xrightarrow[f]{} & S^2 \end{array}$$

Before going to the near-symplectic case, we briefly comment on the symplectic one. A theorem from Thurston tells us, that if g is a compact symplectic fibration and there is a class $\alpha \in H^2(M)$ such that $\iota^*\alpha = [\sigma_b]\,\forall b \in S^2$, where $\sigma_b \in \Omega^2(F_b)$ is the canonical form of the fibre, then M is symplectic. We can pullback this information to W via $\tilde{f}$, and obtain a class $\tilde{\alpha} = \tilde{f}^*\alpha \in H^2(W)$ with the same property. Thus, we only need X^4 to be symplectic in order that W is a symplectic 6-manifold via the induced map $\tilde{g}$. Now we discuss the near-symplectic scenario. The following proposition generates different kind of examples.

Proposition 3.3. *Let $g\colon M^4 \to S^2$ be a compact symplectic fibration with symplectic total space M, (X, ω_X) be a closed, near-symplectic, 4-manifold, and let W be the pullback, defined in the previous parragraph. Then, the pullback W carries a near-symplectic structure induced by $\tilde{g}\colon W \to X$ with singular locus being a surface bundle over S^1.*

Proof. Let Γ be the singular locus of ω_X, that is a disjoint union of circles in X. Its preimage under $\tilde{g}$ is a surface bundle over S^1, and we will denote by Z its total space. This bundle will become the singular locus of the near-symplectic form of W. Let $N\Gamma$ be a tubular neighborhood and let $E = \tilde{g}^{-1}(N\Gamma)$. E is a surface bundle over $S^1 \times D^3$ isomorphic to $Z \times D^3$. We will also consider a smaller subset $\bar{E} \subset E$.

Now we construct a suitable closed 2-form $\tilde{\eta} \in \Omega^2(W)$. Since g is a symplectic fibration, we have suitable class $\alpha \in H^2(M)$. We choose $\tilde{\eta}$ such that $[\tilde{\eta}] = \tilde{f}^*\alpha \in H^2(W)$ with $\iota^*\tilde{\alpha} = \tilde{f}^*[\sigma]$. Secondly, as $\bar{E}$ and E are cohomologically 3-dimensional, we can select $\tilde{\eta}$ with the property that $\tilde{\eta}^2|_{\bar{E}} = 0$.

Let U_k be contractible open subsets of a cover of S^2 with trivializations ϕ_k, such that $\phi_k \circ \phi_j^{-1}$ are symplectomorphisms over $U_k \cap U_j$. We bring these neighborhoods to W as $(\tilde{g} \circ f)^{-1}(U_k) = \tilde{U}_k$. Define $\psi_k := (\text{proj} \circ \phi_k \circ \tilde{f}) : \tilde{U}_k \to F$. Over $\tilde{U}_k$ there is a 1-form μ_k such that $d\mu_k = \psi^*\tilde{\sigma} - \tilde{\eta}$, since $[\tilde{\eta}] = \tilde{f}^*|_F(\alpha) = [\psi^*\tilde{\sigma}]$.

The rest of the proof follows similarly as in step 3 and 4 of theorem's 1 proof. Choose a partition of unity $\rho\colon W \to [0,1]$ in such a way that its open subsets do not touch $\bar{E}$, and with it define a closed 2-form $\hat{\eta} = \tilde{\eta} + \sum_k \rho_k d\mu_k$ on W. This form has the properties that: $\hat{\eta}|_F = \sigma_b$, and $\hat{\eta}|_{\bar{E}} = \tilde{\eta}|_{\bar{E}}$. Finally, we build up our global form by adding $\tilde{g}^*\omega_X$. If K is a sufficiently large positive real number, then we have a closed 2-form, which is non-degenerate away from Z

$$\omega_K = \hat{\eta} + K \cdot \tilde{g}^*\omega_X$$

We conclude by saying a word about the degeneracy of ω_K at Z. At the surface bundle we have that $\omega_K|_Z = \tilde{\eta}|_Z$, thus $\omega_K^2|_Z = 0$. Let σ_b be the canonical symplectic form of the fibre. Denote by ω_F the restriction $\omega_K|_F = \sigma_b$ over every point of the circle $p \in \Gamma \subset X$. Define a closed 2-form $\omega_F \in \Omega^2(D^1 \times F)$, and let $\varphi\colon F \to F$ be a symplectomorphism of the fibre. Consider the mapping torus $(D^1 \times F)\big/((-1, x) \sim (1, \varphi(x)))$ to form a surface bundle Z over S^1. By gluing the

points $(-1, x)$ and $(1, \varphi(x))$ by the symplectomorphism of the surface, we obtain a well-defined closed 2-form ω_Z of rank 2 on the whole bundle. Furthermore, we can find a coordinate neighborhood U around Z such that ω_Z always looks like $h(y_1, y_2)\, dy_1 \wedge dy_2$ for some smooth function h. $\square$

3.2.2. **Near-symplectic 6-manifolds coming from BLFs**

Broken Lefschetz fibrations provide also ways to obtain near-symplectic fibrations on 6-manifolds over near-symplectic 4-manifolds. Let X be symplectic, $g\colon M \to S^2$ be a BLF, and M be a near-symplectic 4-manifold. In particular, in this situation, there is a class $\alpha \in H^2(M)$ such that $\langle \alpha, F \rangle > 0$. Then, W is near-symplectic via a generalized BLF $\tilde{g}$, if $\tilde{\alpha}|_{\Sigma_{\tilde{g}}} = 0 \in H^2(W)$, when restricted to the fold singular set Σ of $\tilde{g}$.

If both $f\colon X \to S^2$ and $g\colon M \to S^2$ are two BLFs, then we require the intersection of their critical images to be transversal in S^2, but not necessarily disjoint. In that case, it follows from standard differential topology that W is a 6-dimensional manifold. The maps $\tilde{f}$ and $\tilde{g}$ become near-symplectic fibrations, carrying the same type and number of fold and Lefschetz-type singularities as f and g respectively. Around a critical point in f^*M, the maps φ and π are locally modelled by coordinate charts

$$\begin{aligned} \tilde{\varphi}\colon \mathbb{R}^6 &\to \mathbb{R}^4 & \qquad \tilde{\pi}\colon \mathbb{R}^6 &\to \mathbb{R}^4 \\ (r, w) &\mapsto (r_1, r_2, r_3, w_1^2 + w_2^2 - w_3^2) & \qquad (r, w) &\mapsto (r_1^2 + r_2^2 - r_3^2, w_1, w_2, w_3) \end{aligned}$$

where $(r, w) = (r_1, r_2, r_3, w_1, w_2, w_3) \in \mathbb{R}^3 \times \mathbb{R}^3$. Assume the cohomological condition on the class $\tilde{\alpha} \in H^2(W)$ as above, and denote by Γ the singular locus of ω_X, and Σ the singularity set of $\tilde{g}$. The mapping $\tilde{g}$ becomes a a near-symplectic fibration over a near-symplectic base (X^4, ω_X), if $\tilde{g}^{-1}(\Gamma) \not\subset \Sigma$ in W. This construction gives 2 generalized BLFs, one for each pullback mapping.

3.2.3. S^6 **over** S^4

As a topological example of the previous maps, we can find a a singular fibration with fold singularities $f\colon S^6 \to S^4$ with one S^3 as singular set of folds of f. Broadly speaking this construction starts by considering $S^6 = S^4 \times B^2 \bigcup B^5 \times S^1$, and then it uses the decomposition of S^4 over S^2 coming from BLFs.

Consider $S^6 = S^4 \times B^2 \bigcup B^5 \times S^1$. Then, using the decomposition of S^4 from the BLF's, and looking at $B^5 \times S^1$ as $B^3 \times B^2 \times S^1$, we have:

$$S^6 = \left(T^2 \times B^4 \cup M_1 \times (S^1 \times B^2) \cup S^2 \times B^4\right) \bigcup B^3 \times B^2 \times S^1$$

where M_1 is the cobordism between T^2 and S^2. In $Y := B^3 \times B^2 \times S^1$ we have another cobordism $M_2 = B^3 - \mathtt{Int}(S^1 \times B^2)$ between the fibres T^2 and S^2, which is glued to M_1 along $S^1 \times S^1$. These two cobordisms run along the two solid tori of S^3. Thus, the fibres can change twice its genus along the singularity set (or four, if we count the reverse direction), but always between S^2 and T^2. Now, $T^2 \times B^4$ gets mapped down to B^4_N and $S^2 \times B^4$ to B^4_S. The middle part $M_1 \times (S^1 \times B^2)$ goes to one of the solid tori of the equator $W = \mathtt{I} \times S^3$, by thinking of S^3 as the union of two

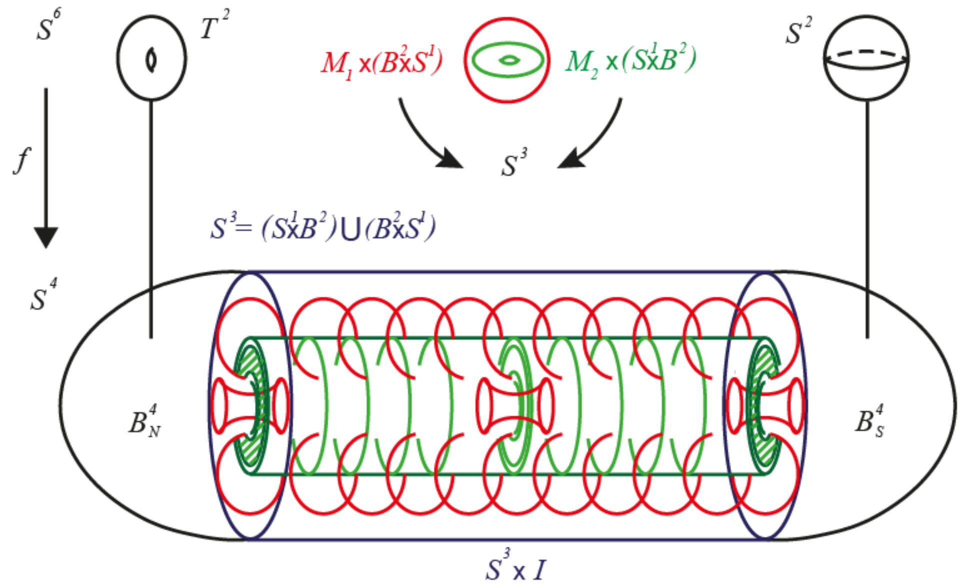

FIGURE 3. Singular fibration of S^6 over S^4 with fold singular locus S^3

solid tori. Y gets mapped to the other solid tori of the equator. Hence, $f^{-1}(B^4_N) = T^2 \times B^4$, $f^{-1}(B^4_S) = S^2 \times B^4$, and above the equator, $f^{-1}(W) = (M_1 \cup M_2) \times S^3$.

This fibration gives us the following picture of S^6 over S^4. The base is divided into 3 parts: northern hemisphere B^4_N, equator $W = S^3 \times I$, and southern hemisphere B^4_S. The fibers over B^4_N of S^4 are 2-torus, and the fibers over B^4_S are 2-spheres. Above the equator occurs the change of fibers. On top of W we have $S^3 \times (M_1 \cup M_2)$, where S^3 is the singularity set of the fibration, and M_1 and M_2 are the standard cobordisms between T^2 and S^2. A schematic picture of this fibration is represented in figure 2, where the S^3 of the base is depicted as the union of two solid tori, one in red and the other one in green.

3.3. **Proof of theorem 1**

First we will construct the local model of the near-symplectic form around the singularity set. The following lemma will be used for this matter. In particular, it will guarantee us that the normal forms of the fold singularity can be pullbacked symplectically.

Lemma 3.4. *Let $p \in U \subset W$ be a point of a submanifold W of codimension 1 embedded in $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$. There is a symplectomorphism $\phi\colon U \subset \mathbb{R}^{2n} \to \tilde{U} \subset \mathbb{R}^{2n}$ mapping U to $\tilde{U} \subset \mathbb{R}^{2n-1} \times 0 \subseteq \mathbb{R}^{2n}$.*

Proof. Let Σ be represented as a graph in $(\mathbb{R}^{2n}, \omega_{st})$, i.e.

$$\Sigma = \{(y_1, \dots, y_{2n-1}, h(y_1, \dots, y_{2n-1})\}$$

We construct now the symplectomorphism $p_1 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ sending the graph Σ to $\mathbb{R}^{2n-1} \times 0$. Define the first coordinate of the smooth map to be the projection to the

$(2n-1)$-th coordinate, $p_1\colon (y_1,\dots,y_{2n}) \mapsto y_{2n-1}$. Its Hamiltonian vector field V_{p_1} can be easily modified to be transverse to the Σ. The second coordinate is defined as

$$q_2\colon (y_1,\dots,y_{2n}) \mapsto (y_{2n} - h(y))$$

Take a point $a = (y_1,\dots,y_{2n-1},h(y)) \in \Sigma$. Move it along the Hamiltonian flow V_{p_1} until it reaches the the point $z = V^t_{p_1}(a) = (y_1,\dots,h(y)-t)$ in time t. Our second coordinate q_1 is a function of the point $z = V^t_{p_1}(a)$ under the action of the Hamiltonian flow $V^t_{p_1}$ measuring the time taken from a to z, thus $q_1((y_1,\dots,y_{2n}) = y_{2n} - h(y) = t$. On the hypersurface we have $y_{2n} = h(y)$, thus $q_1|_\Sigma = 0$, and the Poisson bracket $\{q_1,p_1\} = 1$. The Hamiltonian vector field V_{q_1} lies in $T_a\Sigma$. The remaining two coordinates are: $p_i(y_1,\dots,y_{2n}) \mapsto y_{2i-3}$ and $q_i(y_1,\dots,y_{2n-1}) \mapsto y_{2i-2}$. Since the derivative of p_i in the direction of the Hamiltonian vector field V_{q_i} is 1, then $\{p_i,q_i\} = 1$. Furthermore, V_{p_i} and V_{q_i} do not act in the direction of any of p_1,q_1 or any of the other coordinates. Thus, the Poisson brackets of q_i and p_i vanish with all other coordinates. We get then

$$\{p_i,q_i\} = 1 \quad , \quad \{p_i,q_j\} = 0\,,\,(i \neq j)$$

Hence, $\{p_1,q_1,\dots,p_n,q_n\}$ form a symplectic basis in which $\tilde{\omega} = \sum dp_i \wedge dq_i$. With these coordinates we obtain a symplectomorphism $\phi\colon (\mathbb{R}^{2n},\omega_{st}) \longrightarrow (\mathbb{R}^{2n},\tilde{\omega})$ sending $\Sigma = (y_1,\dots,y_{2n-1},h(y))$ to $\Sigma_0 = (\bar{y}_1,\dots,\bar{y}_{2n-1},0) \simeq \mathbb{R}^{2n-1}\times 0$ $\square$

Constructing the local 2-form

We want to define the local near-symplectic form near the singular sets $\Sigma \sqcup C$, where Σ denotes the singularity set of folds and C of extended Lefschetz-type singularities. First, we define a singular symplectic form vanishing at Σ, and then we pull back the symplectic form of the base. Let $((t = t_1,t_2,t_3),(x = (x_1,x_2,x_3))$ be coordinates around a fold point $p \in \Sigma$ of index 1, locally modeled by $\tilde{f}\colon (t,x) \mapsto (t_1,t_2,t_3,-x_1^2 + \frac{1}{2}(x_2^2+x_3^2)$. Since the fibres are 2-dimensional, we can take a similar local model as the near-symplectic forms on 4-manifolds. Define the following 2-form on a piece of the tubular neighborhood of Σ containing p:

$$\tau_p = d\left(\chi(t)x_1(x_2dx_3 - x_3dx_2)\right) \tag{2}$$

This 2-form is closed, evaluates positive on the fibres, vanishes at the singularity set, and is non-degenerate outside Σ. The map $\chi(t)$ is a smooth cut-off function that will help us in the gluing process when summing up τ_{p_i}'s to build a local 2-form on the whole tubular neighborhood of Σ.

Now, we add the pullback of the symplectic form of the base. Let $\tilde{f}\colon \mathbb{R}^6 \to \mathbb{R}^4$ be the standard fold map. Choose contractible open subsets U_k in a finite cover $\{U_k\}_k$ of the image of Σ in X^4, and let $\varphi_k\colon U_k \to \mathbb{R}^4$ be Darboux charts so that $\left(\varphi_k^{-1}\right)^*\omega_{X^4} = \omega_{\mathbb{R}^4}$. The maps φ_k could modify the local parametrization of the folds coming from $\tilde{f}$ and the position of the critical value set. Nevertheless, this can be fixed. We say now a word of how this can be done.

Start with $f(p) \in X^4$ for $p \in \Sigma$. Choose standard coordinates (y_1,y_2,y_3,y_4) in $\mathbb{R}^4$ near $\tilde{f}(p)$ such that $\omega_{st} = dy_1 \wedge dy_2 + dy_3 \wedge dy_4$. Around a point $p \in \Sigma$, we have $\mathtt{Rank}_f(p) \geq 3$. By the rank theorem we can find coordinates such that $\tilde{f}\colon (t,x) \mapsto (t_1,t_2,t_3,h(t,x))$ To obtain the nice representation of the folds singularities, we need that the image of Σ is described by $h(t,x) = 0$. At the moment, the image

of Σ sits as a graph inside $\mathbb{R}^4$, but not necessarily looking as $\mathbb{R}^3 \times 0$. Denote by C the image of Σ in $\mathbb{R}^4$ moved by φ. By the previous lemma, we can modify C symplectomorphically so that $C \simeq \mathbb{R}^3 \times 0 \subset \mathbb{R}^4$.

$$\begin{tikzcd}
M^6 \supset V \arrow[r, "f"] \arrow[d, "\psi"'] & \tilde{V} \subset X^4 \arrow[d, "\nu"'] & U_k \supset f(\Sigma) \arrow[l, hook', "i"'] \arrow[dl, "\varphi_k"] \\
\mathbb{R}^6 \arrow[r, "\tilde{f}"] \arrow[d, "\zeta"'] & \mathbb{R}^4 \arrow[loop below, "\phi"] & \\
\mathbb{R}^6 \arrow[ur, "\pi"'] & &
\end{tikzcd}$$

The line of reasoning is now a fold version of the Morse Lemma [GG73]. The restriction $f|_\Sigma \colon \Sigma \to (t_1, t_2, t_3, 0)$ is a local diffeomorphism near p. Thus, we can choose coordinates in the domain so that Σ is defined near p by $x_1 = x_2 = x_3 = 0$. By the properties of the singular set, Σ is also described by $\frac{\partial h}{\partial x_i}(0) = 0$, and $\frac{\partial^2 h}{\partial x_i^2}(0) \neq 0$. After applying the symplectomorphism ϕ on $\mathbb{R}^4$, C is described by $y_4 = 0$. Thus, we get $h(t,x) = 0$, when $x_1 = x_2 = x_3 = 0$, and we can express h as

$$h(t,x) = \sum_{1 \leq i,j \leq 3} b_{ij}(x) x_i x_j$$

where $b_{ij}(x)$ are smooth functions. Moreover, the (3×3)-matrix $b_{ij}(0)$ is nonsingular. This means that at f_4 we are in a situation of a function with a non-degenerate critical point. We only need to perform the remaining coordinate changes in the domain $\zeta \colon \mathbb{R}^6 \to \mathbb{R}^6$ to obtain the representation of the folds. For the sake of clarity, we can assume that $\tilde{f}$ has this nice parametrization, while preserving the symplectic structure. The pull back $\psi^*\left(\tilde{f}^*\left((\varphi_k^{-1})^*\omega_{X^4}\right)\right) = f^*\omega_{X^4}$ looks locally like $\tilde{f}^*\omega_{\mathbb{R}^4}$, and drops to rank 2 at Σ. By summing up the forms τ_{p_k} over a finite cover of Σ together with the pullback, we obtain the local model

$$\omega_a = \sum_{p_k} \tau_{p_k} + \psi^*\left(\tilde{f}^*\left((\varphi_k^{-1})^*\omega_{X^4}\right)\right) = \tau + f^*\omega_{X^4} \tag{3}$$

This 2-form defines the near-symplectic form on the tubular neighborhood of Σ. It is closed, non-degenerate outside Σ, positive on the fibres, and degenerates to $\mathtt{Rank}(\omega_p) = 2 \ \forall p \in \Sigma$.

Around the elements of C, where f is given by $(z_1, z_2, z_3) \mapsto (z_1, z_2^2 + z_3^2)$, we can choose disjoint neighborhoods B_k such that $\omega_0|_{B_k} = \omega_{\mathbb{C}^3}$. For any $v_1, v_2 \in T_pF$, we get $\omega_0|_{B_k}(v_1, v_2) > 0$ away from the singularity. The symplectic form $\omega_0|_{B_k}$ can be extended to the fibre F_q as a symplectic form for all $q \in f(B_k) \subset X$.

Extension over the neighborhoods of the fibres

Now we extend the local model over the neighborhood of the fibres, such that it agrees with ω_a near $\Sigma \sqcup C$. Let U be the tubular neighborhood of $\Sigma \sqcup C$. Choose a closed 2-form $\bar{\tau} \in \Omega^2(M)$ with a class being represented by α. Since $\alpha|_\Sigma = 0 \in H^2(\Sigma)$, over U there exists a 1-form $\bar{\mu} \in \Omega^1(U)$, such that $\omega_a - \bar{\tau} = d\bar{\mu}$. We extend now $\bar{\mu}$ to an arbitrary 1-form on the manifold, $\mu \in \Omega^1(M)$, supported in a neighborhood W of U. By substituting $\tau = \bar{\tau} + d\mu$ on U, we can regard τ to be ω_a when restricted to U.

For any $q \in X^4$ we can find a tubular neighborhood V_q of the fibre F_q and neighborhoods $U_2 \subset U_1 \subset U$ of the fold singularity set Σ, such that the neighborhoods are sufficiently small and $V_q \cap U_1$ retracts to $F_q \cap \Sigma$. If q is a regular then we can reduce the neighborhoods such that $V_q \cap U_1 = \varnothing$, since $F_q \cap \Sigma = \varnothing$. If q is critical, then we can select them small enough in order to retract them to $F_q \cap \Sigma$. A regular $q \in X$ can be engulfed by a 4-disk. For a fibre F_q, take $f^{-1}(D^4) = V_q$ with a disk small enough. Then by the previous argument $V_q \approx D^4 \times F_q$. If $q \in X$ is singular, then we only throw away the part of the fibre intersecting the neighborhood of Σ, thus $V_q \setminus (V_q \cap U_2) \approx D^3 \times (F_q \setminus (F_q \cap U_2))$.

Finally, we can construct a smooth map $\pi\colon V_q \to V_q$ by using the second property with a projection map $\mathtt{pr}\colon V_q \setminus (V_q \cap U_2) \to F_q \setminus F_q \cap U_2$, and interpolating it with the identity map $id_{V_q \cap U_1}$ with: $\mathtt{Im}(\pi) \subset F_q \cup (V_q \cap U_1)$ and $\pi|_{F_q \cup (V_q \cap U_2)} = id_{F_q \cup (V_q \cap U_2)}$. By assumption $\langle \alpha, F \rangle > 0$ over each component of the fibre, $[\tau] = \alpha$, and the fibres have a symplectic form. Thus, we can equip the fibres with a near-symplectic form σ_q with

(a) $\sigma_q|_{F_q \cap U_1} = \tau$, that is, restricted to U, σ is near-symplectic, since $\tau|_U = \omega_a$. The form σ_q is defined on the fibre, so $\sigma_q|_{F_q \cap U_1}$ is near-symplectic.
(b) $\sigma_q|_{F_q}$ is positive over the smooth part of F_q, since ω_a takes care of the bottom part close to the singularities and the symplectic form on the rest.
(c) $\int_F \sigma_q = \langle \alpha, F \rangle > 0$, since $[\sigma_q - \tau|_{F_q}] = 0$ in $H^2(F_q, F_q \cap U_1) \overset{PD}{\simeq} H_0(F_q, F_q \cap U_1) \simeq 0$ (assuming F_q connected) , then $(\sigma_q - \tau|_{T_q})$ is exact in $F_q \cap U_1$, that is $[\sigma_q] = [\tau] = \alpha$.

We use $\pi\colon V_q \to F_q \cup (V_q \cap U_1)$ to construct a near-symplectic form $\tilde{\tau}$ on V_q with the following features

$$\tilde{\tau} = \pi^* \sigma_q + \pi^* \tau$$

(1) $d\tilde{\tau} = 0$ and $[\tilde{\tau}] = \alpha|_{V_q}$
(2) $\tilde{\tau}|_{V_q \cap U_2} = \tau$
(3) $\exists$ a 1-form μ_q on V_q, $\mu_q|_{V_q \cap U_2} = 0$, such that $\tilde{\tau} - \tau = d\mu_q$, since $[\tilde{\tau} - \tau] = 0$ in $H^2(V_q, V_q \cap U_2) \simeq H^2(F_q, F_q \cap U_2)$. Thus,
$$\tilde{\tau}_q = \tau + d\mu_q$$
(4) $\tilde{\tau}_q|_{F_q} > 0$ restricts positively to the fibre for every regular point $q \in V_q$.

Patching into a global form
We expand the near-symplectic form over the whole manifold M^6. Since our base is compact, we can find a finite subset $Q \subset X^4$ and choose a finite cover $\mathcal{D}$ with open subsets $(D_q)_{q \in Q}$, such that $f^{-1}(D_q) \subset V_q$ for each $q \in X^4$. Consider a smooth partition of unity $\rho\colon X \to [0,1]$, $\sum_{q \in Q} \rho_q = 1$, subordinate to the cover $\mathcal{D}$ with $\mathtt{supp}(\rho_q) \subset D_q$. We build the global 2-form by patching the local 1-forms μ_q previously defined on V_q. Thus, we define the following closed 2-form

$$\hat{\tau} = \tau + d\left(\sum_{q \in Q} (\rho_q \circ f)\mu_q\right) \tag{4}$$

Since f is constant on the fibres, the 1-form $d((\rho_q \circ f)\mu_q) = 0$ when evaluated on the vectors tangent to the fibre. From the second step, τ agrees with ω_a when restricted

to U. Thus, the first summand takes care of the part near the critical set and the second one of the regular part. Let $\bar{U}$ be the intersection of all neighborhoods U_2 for all $q \in Q$, i.e. $\bar{U} = U_2 \bigcap_{q\in Q} f^{-1}(D_q)$. The global form $\hat{\tau}$ becomes τ when restricted to $\tilde{U}$, so it agrees with the local model of ω_a at U_2. Thus, $\hat{\tau}$ is globally well-defined over M^6.

The 2-form $\hat{\tau}$ restricts to a fibre F_q in the following way
$\hat{\tau}|_{F_p} = \tau|_{F_p} + \sum_{q\in Q} \rho \circ f(p) d\mu_q|_{F_p} = \sum_{q\in Q} \rho \circ f(p)(\tau + d\mu_q)|_{F_q}$
$= \sum_{q\in Q} (\rho \circ f(p))\, \tilde{\tau}_q|_{F_p}$

This is a convex combination of near-symplectic 2-forms. On each fibre, $\tilde{\tau}$ is closed, positive outside the singular locus, and vanishes at Σ, inducing a near-symplectic structure on each fibre.

Positivity on vertical and horizontal tangent subspaces
To conclude the global construction, we can apply a similar argument as in the symplectic case [Thu76]. The 2-form τ is positive on the vertical tangent subspaces to the fibre $\mathrm{Ver}_p = \ker df(p) = T_pF \subset T_pM$, outside the singularity set. For a sufficiently large K, the following 2-form is also positive on the horizontal spaces

$$\omega_K = \tilde{\tau} + K \cdot f^*\omega_{X^4} \tag{5}$$

If we restrict ω_K to the vertical tangent subspaces to the fibre, it agrees with $\tilde{\tau}$. Hence, we have obtained a closed 2-form, positive on the fibres, non-degenerate outside Σ, and degenerating to rank 2 along the singularity set Σ. □

Remark. Even though, lemma 3.5 together with step 1 tell us that the fold map pulls back symplectomorphically, other types of singularities might need a different treatment. For instance, if we would like to consider deformations of near-symplectic fibrations, in a similar fashion as Lekili [Lek09], then it would be necessary to consider all stable singularities of maps from $\mathbb{R}^6$ to $\mathbb{R}^4$: folds, cusps, swallowtails, and butterflies.

4. Contact structure induced by a near-symplectic form

4.1. Proof of proposition 1

The proof follows a similar argument as in the standard symplectic case. However, we need to make some adjustments due to the near-symplecticity. Consider the fold map of a generalized BLF parametrized by

$$f\colon T^*Z \simeq Z^3 \times \mathbb{R}^3 \to Z^3 \times \mathbb{R}$$
$$(z,x) \mapsto (z, -x_1^2 + \frac{1}{2}(x_2^2 + x_3^2))$$

To obtain the shape of the near-symplectic form on the cotangent bundle, we pullback the symplectic form $\omega_B = d(e^t\alpha_Z)$ with the previous map and add the 2-form $\tau = e^c d(x_1(x_2dx_3 - x_3dx_2))$ in a similar fashion as in equation 2 . This gives us the following near-symplectic form on $Z \times \mathbb{R}^3$

$$\omega_{\text{ns}} = e^c\{dz_1 \wedge dz_2 + 2x_1(dz_3 \wedge dx_1 + dx_2 \wedge dx_3) - x_2(dz_3 \wedge dx_2 - dx_1 \wedge dx_3)$$
$$- x_3(dz_3 \wedge dx_3 + dx_1 \wedge dx_3) + \rho$$

Here, the factor on e is $c = -x_1^2 + \frac{1}{2}(x_2^2 + x_3^2)$. We treat the last term ρ as noise, but we point out that it has no qualitative impact in the construction of the contact structure. For completeness, $\rho = z_1(2x_1 dz_2 \wedge dx_1 - x_2 dz_2 \wedge dx_2 - x_3 dz_2 \wedge dx_3)$. We give global coordinates (z, x) to the bundle $E = T^*Z$, identifying with $z = (z_1, z_2, z_3)$ the coordinates on Z and with $x = (x_1, x_2, x_3)$ the ones in the normal direction. Take the natural radial vector field on the cotangent bundle (in the symplectic case it is a Liouville vector field) $\tilde{Y} \colon T^*Z \to TE, (z, x) \mapsto (0, x)$, and isotope it to obtain a vector field with the same qualitative properties

$$Y = \frac{1}{e^{\rho}} \left(\frac{1}{2} x_1 \left(\frac{\partial}{\partial x_1} \right) + x_2 \left(\frac{\partial}{\partial x_1} \right) + x_3 \left(\frac{\partial}{\partial x_1} \right) \right)$$

Insert this vector field into the second slot of the near-symplectic form to obtain the 1-form

$$\alpha_{\mathrm{N}} := \iota_Y \omega_{\mathrm{ns}} = (x_1^2 - x_2^2 - x_3^2) dz_3 + \frac{5}{2} x_1 (x_3 dx_2 - x_2 dx_3) + \beta \tag{6}$$

where $\beta = \iota_Y \rho = z_1(x_1^2 - x_2^2 - x_3^2) dz_2$. Notice that neither Y nor $\tilde{Y}$ is a Liouville vector field for the near-symplectic form. This is due to the degeneracy of ω_{ns}. Thus, we pull back α_Z, the defining 1-form of some contact structure ξ_Z on the 3-submanifold Z using the natural projection $\pi \colon ST^*Z \simeq Z \times S^2 \to Z$. We use a Darboux chart on α_Z for this purpose. Let $K > 0$ be a real number. Then, the contact structure $\xi_{Z \times S^2} = \ker(\alpha)$ is defined by the contact form:

$$\alpha = K \cdot \alpha_N - \pi^* \alpha_Z \tag{7}$$

With an easy computation we can check that the contact condition is satisfied on $ST^*Z \simeq Z \times S^2$ for a sufficiently large K

$$\alpha \wedge d\alpha^2 = 2K^2 \alpha_N \wedge d\alpha_N \wedge \pi^* d\alpha_Z - 2K \pi^*(\alpha_Z \wedge d\alpha_Z) \wedge d\alpha_N > 0 \qquad \square$$

4.2. **Interaction of ω_0 with a contact structure ξ_Z at the singular locus**

Consider the class of examples generated by the pullback bundle via $\tilde{g}$, where $(X, \omega_{\mathrm{ns}})$ is near-symplectic, closed 4-manifold, and g is a symplectic fibration. Recall that the singular set $Z = \tilde{g}^{-1}(\Gamma)$ of ω_W is a surface bundle over S^1. Let S_g be the closed oriented surface representing the fibre of the singular set in W. We can construct a mapping torus of Z via an orientation preserving diffeomorphism of the fibre $\phi \colon S_g \to S_g$

$$S_g(\phi) = S_g \times D^1 / (x, 0) \sim (\phi(x), 2\pi)$$

Remove a small disk D^2 from the surface. Choose an orientation preserving diffeomorphism ϕ_1 of $\bar{S}_g = S_g \setminus D^2$, such that ϕ_1 leaves a disk fixed. Put together the mapping torus $\bar{S}_g(\bar{\phi_1})$. Similarly, make one $D^2(\phi_2)$ of the extracted disk. We build a contact structure induced by the near-symplectic form ω_{ns} on both mapping tori. By gluing them back together, we can obtain a contact structure on a mapping torus diffeomorphic to the original one, as every orientation preserving diffeomorphism is isotopic to one leaving a disk fixed.

As the class of the near-symplectic form restricts to the canonical fibre class at the singular loucs Z, we can assume that ω_0 is an area form of total area 2π. Since $\int_{\bar{S}_g} \omega_0 - d\beta_0 = 0$, it follows that $(\omega_0 - d\beta_0) = 0 \in H^2_c(\bar{S}_g)$. Thus, we can find a compactly supported 1-form β_1 with support inside $\bar{S}_g \setminus \left([-\frac{3}{2}, 0] \times \partial \bar{S}_g\right)$ and

$(\omega_0 - d\beta_0) = d\beta_1$. Let's say that its compact support finishes at $[-\frac{3}{2}, 0] \times \partial \bar{S}_g$. Let $\beta := \beta_0 + \beta_1$, and define on $\bar{S}_g(\bar{\phi})$ the following 1-form

$$\bar{\beta} = \mu(r)\beta + (1 - \mu(r))\bar{\phi_1}^*\beta$$

where μ is a smooth step-up function defined on D^1 taking values on $[0,1]$, such that $\mu = 1$ near $r = 0$ and $\mu = 0$ near $r = 2\pi$. On each fibre F of $\bar{S}_g(\bar{\phi})$ over $t \in S^1$, this 1-form has the following properties: (i) $\bar{\beta}|_F = e^s d\theta$ on the collar $[-\frac{3}{2}, 0] \times \partial \bar{S}_g$ and (ii) $d\bar{\beta}|_F = \omega_0$ with $\int_{\bar{S}_g} d\beta_0 = 2\pi$. The 1-form

$$\alpha_1 = \bar{\beta} + K \cdot dt \tag{8}$$

defines a contact form on the mapping torus by choosing a sufficiently large real constant $K > 0$. We can apply the same argument to obtain a contact form α_2 on the other mapping torus $D^2(\varphi)$. By gluing both mapping tori we obtain a well-define contact form α on the original mapping torus $\bar{S}_g(\phi)$.

5. PS-overtwistedness of the $(Z \times S^2, \xi)$

5.1. Notions of overtwistedness

In dimension 3 contact structures are classified in *tight* and *overtwisted*. A contact structure is called *overtwisted* if (M^3, ξ) contains an embedding of a disk $D^2 \hookrightarrow M^3$ such that for its characteristic foliation: (i) there is a unique elliptic singular point in the interior, and (ii) the boundary ∂D is a closed leaf. If we cannot find such an

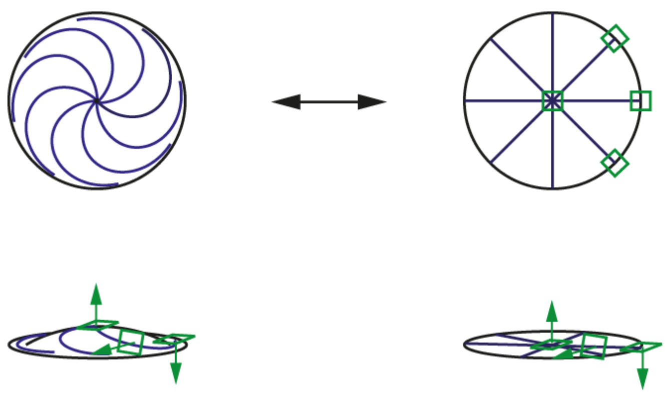

FIGURE 4. Two equivalent representation of the overtwisted disk. Left: the disk has a bump. Right: the disk is flat.

overtwisted disk, then the contact structure is called *tight*.

In higher dimensions, Niederkrüger introduced the concept of *plastikstufe*, which generalizes the idea of an overtwisted disk [Nie06]. Recently, Massot, Niederkrüger, and Wendl gave a generalization of the plastikstufe with an object called *bLob* [MNW12]. Before going to higher dimensions, we recall one more concept of 3-manifolds, the one of the Giroux torsion. Its analogue in higher dimensions is the Giroux domain.

Definition 5.1. A *half-torsion domain* is a thickened torus $[0,1] \times T^2 \ni (r,(x,y))$ with contact structure $\xi = \ker\{\sin(\pi r)dx + \cos(\pi r)dy\}$.

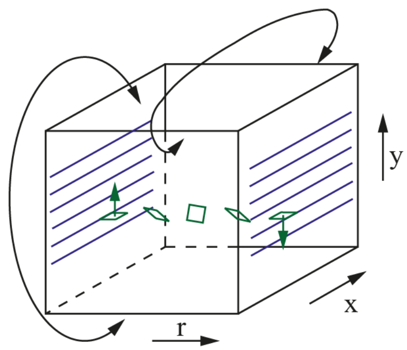

FIGURE 5. Half-Torsion Domain

The half-torsion domain can be represented as a cube. Since (x,y) are the torus coordinates, we identify the top and bottom faces, as well as the front and back ones. The contact structure is horizontal on the left and right faces, and the r-direction is always tangent to ξ. Taking any line-r, we obtain a half twist of the contact plane. This happens everywhere, whatever line we choose.

Definition 5.2. (M^3,ξ) has *positive Giroux torsion*, if it contains two half-torsion domains glued along one face. This means that along a line-r, inside the two blocks, the contact plane makes a full turn from side to side. If (M^3,ξ) is overtwisted, then it has positive torsion.

Now we consider the situation in higher dimensions.

Definition 5.3. A contact manifold (V^{2n+1},ξ) is called *PS-overwtisted* if it contains a submanifold $N \subset B^{2n+1} \subset V$ with the properties:

(1) $\dim(N) = n+1$, where N is compact with boundary
(2) $\xi \cap TN$ induces a singular foliation that looks like an open book
(3) ∂N lies in the singular set of the foliation

The binding B of the open book is a codimension 2 submanifold in the interior of N with trivial normal bundle. In addition, $\theta\colon N \setminus B \to S^1$ is a fibration whose fibers are transverse to the boundary of N, and which coincides in a neighborhood $B \times D^2$ of $B = B \times \{0\}$ with the normal angular coordinate.

These objects are also called *bordered Legendrian open books* or *bLobs*. A more precise definition appears in [MNW12].

In dimension 3 a flat overtwisted disk D_{ot} is a bLob. Another simple example of a bLob comes by considering a closed manifold C and then taking the product $N = C \times D^2_{\text{ot}}$ with structure pulled back from the overtwisted disk.

Definition 5.4. Let (V,ξ) be a contact manifold and H an oriented hypersurface. H is called a *ξ-round hypersurface modeled on some closed manifold Z* if it is transverse to ξ and admits an orientation preserving identification with $S^1 \times Z$ such that $\xi \cap TH = TS^1 \oplus \xi_Z$.

Definition 5.5. Let Σ be a compact manifold with boundary, ω a symplectic form on the interior of Σ and ξ a contact structure on the boundary of Σ. The triple (Σ,ω,ξ) is an *ideal Liouville domain* if there exists an auxiliary 1-form β on Int(Σ) such that:

- $d\beta = \omega$ on Int(Σ), and
- For any smooth function $f\colon \Sigma \to [0,\infty)$ with regular level set $\partial\Sigma = f^{-1}(0)$, the 1-form $f\beta$ extends smoothly to $\partial\Sigma$ such that its restriction to $\partial\Sigma$ is a contact form on ξ.

We will refer to $\Sigma \times S^1$ with the contact structure $\xi = \ker(\beta + dt)$ as the *Giroux domain* associated to (Σ,ω,ξ).

Now we cite a theorem from Massot, Niederkrüger and Wendl that tells us that gluing and blowing down Giroux domains gives us a PS-overtwisted contact structure.

Theorem 5.6 ([MNW12])**.** *Let (V,ξ) be a closed contact manifold containing a subdomain N with nonempty boundary, which is obtained by gluing and blowing down Giroux domains. If N has at least one blown down boundary component, then it contains a small* bLob*, hence (V,ξ) is PS-overtwisted.*

5.2. **Proof of theorem 2**

Proof. The contact structure on $Z \times S^2$ is given by the kernel of $\alpha = K\cdot\alpha_{\mathrm{N}} - \pi^*\alpha_Z$, as constructed in 4.1, where

$$\alpha_{\mathrm{N}} := \iota_Y\omega_{\mathrm{ns}} = (x_1^2 - x_2^2 - x_3^2)dz_3 + \frac{5}{2}x_1(x_3dx_2 - x_2dx_3) + z_1(x_1^2 - x_2^2 - x_3^2)dz_2$$

Consider now the S^1–action

$$(z_1,z_2,z_3,x_1,x_2,x_3) \mapsto (z_1,z_2,z_3,x_1,x_2\cos\varphi - x_3\sin\varphi, x_2\sin\varphi + x_3\cos\varphi)$$

acting on $(Z\times S^2,\xi)$ and the vector field $X_{S^1} = \frac{d}{d\varphi}\Big|_{\varphi=0} = -x_3\left(\frac{\partial}{\partial x_2}\right) + x_2\left(\frac{\partial}{\partial x_3}\right)$. The fixed point set obtained from this action is:

$$P_{S^1} = \{(z_1,z_2,z_3,\pm1,0,0\} = Z^3 \times \{\pm1\} \tag{9}$$

This submanifold of codimension 2 has trivial normal bundle as shown in Lemma 4.2, and it will become the the binding of the *bLob*. Let γ_{eq} denote the equator of S^2. For the dividing set we get

$$\Gamma_{X_{S^1}} = \{p \in ST^*Z \mid \alpha(X_{S^1}) = 0\} = Z \times (\gamma_{\text{eq}} \sqcup \{\pm1\}) \tag{10}$$

since $\alpha(X_{S^1}) = \frac{5}{2}x_1(x_x^2 + x_3^2) = 0 \iff x_1 = 0,\ x_2 = x_3 = 0$. The equator of S^2 and the north and south poles are in the S^2-fibres of the bundle. From this, we can read that there is decomposition of ST^*Z in two parts, $Z\times D_+^2$ corresponding to $\{x_1 \geq 0\}$ and $Z \times D_-^2$ corresponding to $\{x_1 \leq 0\}$. The S^1-action leaves the

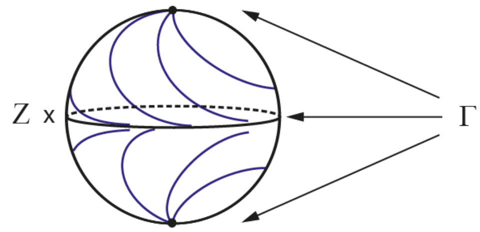

FIGURE 6. Dividing set of $Z \times S^2$

boundary invariant, and on the boundary it is tangent to the contact structure. Each of these components has a submanifold of fixed point sets of codimension 2 with trivial normal bundle.

Thus, $(Z \times S^2, \xi)$ is composed by two Giroux domains $\Sigma_\pm \times S^1$. The compact hypersurfaces $\Sigma_+ = Z \times [0,1]$ and $\Sigma_- = Z \times [-1,0]$ are ideal Liouville domains. We glue the two Giroux domains domains along the common boundary, and collapse the other two that remained free. We explain now how this process works.

Consider the boundaries $W_1^+ = Z \times \{1\} \times S^1$ and also $W_1^- = Z \times \{-1\} \times S^1$. These two boundaries correspond to a ξ-round hypersurface in the *bLob* context. By [MNW12], we can find a neighborhood around each of them contactomorphic to

$$\big(Z \times [0,\varepsilon) \times S^1\,,\, \alpha_Z + r\,d\theta\big)$$

Now comes the blow down. We substitute $Z \times [0,\varepsilon) \times S^1$ by $Z \times D^2$ using a diffeomorphism with polar coordinates

$$\psi\colon Z \times D^2 \to Z \times [0,\varepsilon) \times S^1$$
$$(z,r,\theta) \mapsto (z,r^2,\theta)$$

which pulls back the contact form to $\alpha_Z + r^2 d\theta$. This gives us a good contact form on $Z \times \{0\}$ and a blow down of the top and bottom boundaries of the Giroux domaIns. To glue the remaining two boundaries W_0^+, W_0^- we take again two boundaries, both of which will be contactomorphic to

$$(Z \times (-\varepsilon, 0] \times S^1, \alpha_Z + r\,d\theta)$$

we can glue them by extending both neighborhoods to $Z \times (-\varepsilon,\varepsilon) \times S^1$ and gluing with a diffeomorphism $\varphi\colon (z,r,\theta) \longmapsto (z,-r,-\theta)$. Now, we recall a result by Massot, Niederkrüger and Wendl [MNW12, Theorem 5.13] 5.6. In our situation we have actually 2 Giroux domains that have been glued and blown down, concluding that $(Z \times S^2, \xi)$ is PS-overtwisted.

Now we show the statement related to the contact fibration. Consider the same contact structure on $Z \times S^2$ defined by α as in equation 7. Take the natural projection $\pi\colon Z \times S^2 \to S^2$ and the inclusion $\iota\colon Z \hookrightarrow Z \times S^2$. If we pull back the contact form α into the fibre Z, we obtain

$$\iota^*\alpha = (K \cdot B - 1) z_1 dz_2 + dz_3$$

As in 7 , K is a sufficiently large positive constant. The factor B is a smooth surjective function $B\colon S^2 \to [-1,1]$, which at a point $p_0 \in S^2$ is defined by $B(x_1,x_2,x_3)_{p_0} = -x^2_{1,p_0} + x^2_{2,p_0} + x^2_{3,p_0}$. The 1-form $\iota^*\alpha$ vanishes whenever $B = \frac{1}{K}$, and that is the set where the fibres failed to be of contact type. If $B = \frac{1}{K}$ is a regular value of B, then $C := B^{-1}\left(\frac{1}{K}\right)$ is a 1-submanifold of S^2. Outside this subset C, we have a contact fibration $\pi\colon Z \times S^2 \setminus C \to S^2 \setminus C$ with contact fibres. □

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